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SOLUTION OF A HEAT-CONDUCTION PROBLEM FOR A FINITE CYLINDER AND SEMISPACE UNDER MIXED LOCAL BOUNDARY CONDITIONS IN THE PLANE OF THEIR CONTACT

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With the use of the method of summator–integral equations, an axisymmetric problem has been investigated that deals with the development of spatial temperature fields appearing in a finite cylinder with an arbitrary distribution of initial temperature when the cylinder comes in contact with a semi-infinite body that has a constant initial temperature. The essential feature of the considered thermophysical model of heat exchange is that mixed boundary conditions of the second and fourth kind are assigned in the plane of contact of the finite body with the semispace. The thermophysical properties of the bodies considered are different.

The formulation of the problem consists of the determination of the laws governing the development of spatial nonstationary temperature fields in a semispace and a finite cylinder of radius R and height l, when one of the end faces of the finite cylinder touches the semispace surface. The thermophysical characteristics of the bodies considered and their initial temperatures are different and the distribution of the initial temperature of the cylinder is arbitrary. Outside the circular region of contact on the surface of the semispace and on the lateral and noncontacting end-face surface of the cylinder there is ideal thermal insulation. Thereafter, when writing down mathematical formulas, the subscript 1 is used for the semispace and 2 for the cylinder.

Let r and z denote cylindrical coordinates, τ time, $T_1(r, z, \tau)$ the temperature of the semi-infinite body ($r > 0$, $z < 0$. $\tau > 0$), $T_2(r, z, \tau)$ the temperature of the cylinder ($0 < r < R$, $0 < z < l$, $\tau > 0$), and $\lambda_1 > 0$ and $a_1 > 0$ and $\lambda_2 > 0$ and $a_2 > 0$ the thermal conductivity and thermal diffusivity of the semi-infinite body and cylinder, respectively.

Thus, it is necessary to solve a system of two differential equations of nonstationary heat conduction (in the corresponding ranges of coordinates)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_1(r, z, \tau)}{\partial r}\right)+\frac{\partial^2 T_1(r, z, \tau)}{\partial z^2}=\frac{1}{a_1}\frac{\partial T_1(r, z, \tau)}{\partial \tau}, \quad r>0, \quad z<0, \quad \tau>0, \tag{1}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_2(r, z, \tau)}{\partial r}\right)+\frac{\partial^2 T_2(r, z, \tau)}{\partial z^2}=\frac{1}{a_2}\frac{\partial T_2(r, z, \tau)}{\partial \tau}, \quad 0<r<R, \quad 0<z<l, \quad \tau>0, \tag{2}$$

with the initial

$$T_1(r, z, 0)=T_{10}=\text{const}, \quad T_2(r, z, 0)=f(r, z)\neq T_{10} \tag{3}$$

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and boundary-value conditions

$$\frac{\partial T_1(r,-\infty,\tau)}{\partial z}=\frac{\partial T_1(0,z,\tau)}{\partial r}=\frac{\partial T_1(\infty,z,\tau)}{\partial r}=\frac{\partial T_2(0,z,\tau)}{\partial r}=\frac{\partial T_2(R,z,\tau)}{\partial r}=\frac{\partial T_2(r,l,\tau)}{\partial z}=0\,, \tag{4}$$

$$T_1(r,0,\tau)=T_2(r,0,\tau)\,,\quad 0<r<R\,, \tag{5}$$

$$K_\lambda\frac{\partial T_1(r,0,\tau)}{\partial z}=\frac{\partial T_2(r,0,\tau)}{\partial z}\,,\quad 0<r<R\,, \tag{6}$$

$$\frac{\partial T_1(r,0,\tau)}{\partial z}=0\,,\quad R<r<\infty\,, \tag{7}$$

where $K_\lambda = \lambda_1/\lambda_2$.

We note that, according to [1], expressions (5) and (6) determine the boundary condition of the fourth kind in the region $z = 0$, $0 < r < R$, and the system of expressions (5)–(7) determines the mixed boundary conditions on the surface $z = 0$ in the corresponding regions of change of the variable r.

To solve the problem formulated, we first apply the Laplace integral transformation (L-transformation) [2], as a result of which, with account for the initial conditions, the problem at hand in the region of transforms will be written as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{T}_1(r,z,s)}{\partial r}\right)+\frac{\partial^2\bar{T}_1(r,z,s)}{\partial z^2}-\frac{s}{a_1}\bar{T}_1(r,z,s)+\frac{T_{10}}{a_1}=0\,,\quad r>0\,,\quad z<0\,, \tag{8}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{T}_2(r,z,s)}{\partial r}\right)+\frac{\partial^2\bar{T}_2(r,z,s)}{\partial z^2}-\frac{s}{a_2}\bar{T}_2(r,z,s)+\frac{f(r,z)}{a_2}=0\,,\quad 0<r<R\,,\quad 0<z<l\,, \tag{9}$$

$$\frac{\partial \bar{T}_1(r,-\infty,s)}{\partial z}=\frac{\partial \bar{T}_1(0,z,s)}{\partial r}=\frac{\partial \bar{T}_1(\infty,z,s)}{\partial r}=\frac{\partial \bar{T}_2(0,z,s)}{\partial r}=\frac{\partial \bar{T}_2(R,z,s)}{\partial r}=\frac{\partial \bar{T}_2(r,l,s)}{\partial z}=0\,, \tag{10}$$

$$\bar{T}_1(r,0,s)=\bar{T}_2(r,0,s)\,,\quad 0<r<R\,, \tag{11}$$

$$K_\lambda\frac{\partial \bar{T}_1(r,0,s)}{\partial z}=\frac{\partial \bar{T}_2(r,0,s)}{\partial z}\,,\quad 0<r<R\,, \tag{12}$$

$$\frac{\partial \bar{T}_1(r,0,s)}{\partial z}=0\,,\quad R<r<\infty\,, \tag{13}$$

in which

$$\overline{T}_i (r, z, s) = L [T_i (r, z, \tau)] = \int_0^\infty T_i (r, z, \tau) \exp (-s\tau)\, d\tau , \quad i = 1, 2 , \tag{14}$$

and the restriction Re $s > 0$ on the parameter of Laplace transformation (L-parameter) is omitted for brevity here and hereafter.

Solution of Eq. (8) under the corresponding conditions from (10) is known [3]:

$$\overline{T}_1 (r, z, s) = \frac{T_{10}}{s} + \int_0^\infty \overline{C} (\rho, s) \exp \left(- \left| z \right| \sqrt{\rho^2 + \frac{s}{a_1}} \right) J_0 (\rho r)\, \rho d\rho , \quad r > 0 , \quad z < 0 , \tag{15}$$

Here $J_0(\rho r)$ is the Bessel function of the first kind and zero order and $\overline{C}(\rho, s)$ is the unknown analytical function.

To solve Eq. (9), we will apply to it the Hankel finite integral transformation [2]

$$\overline{T}_{2H} (p, z, s) = H [\overline{T}_2 (r, z, s)] = \int_0^R r \overline{T}_2 (r, z, s)\, J_0 (pr)\, dr . \tag{16}$$

Then Eq. (9), with account for the corresponding conditions from (10), is transformed into the equation

$$\frac{\partial^2 \overline{T}_{2H} (p, z, s)}{\partial z^2} - \psi^2 (p, s)\, \overline{T}_{2H} (p, z, s) + F (p, z) = 0 , \quad 0 < z < l , \tag{17}$$

in which

$$F (p, z) = \frac{1}{a_2} \int_0^R r f (r, z)\, J_0 (pr)\, dr , \tag{18}$$

$\psi(p, s) = \sqrt{p^2 + s/a_2}$, and $pR = \mu$ are zeros of the Bessel function $J_1(\mu)$.

The solution of differential equation (17) can be easily obtained by the classical method of varying an arbitrary constant (see, e.g., [14]):

$$\overline{T}_{2H} (p, z, s) = \overline{A}_1 (p, s) \cosh [z\psi (p, s)] + \overline{A}_2 (p, s) \sinh [z\psi (p, s)] -$$

$$- \frac{1}{\psi (p, s)} \int_0^z F (p, \xi) \sinh [(z - \xi)\, \psi (p, s)]\, d\xi , \tag{19}$$

Here the unknown functions-transforms $\overline{A}_i(p, s)$ are determined by the boundary condition

$$\frac{\partial \overline{T}_{2H} (p, l, s)}{\partial z} = 0 . \tag{20}$$

Substituting Eq. (19) into Eq. (20), we obtain the value

$$\overline{A}_1 (p, s) = \frac{1}{\psi (p, s) \sinh [l\psi (p, s)]} \int_0^l F (p, \xi) \cosh [(l - \xi)\, \psi (p, s)]\, d\xi - \overline{A}_2 (p, s) \cot [l\psi (p, s)] ,$$

the substitution of which into Eq. (19) allows us to write the following expression:

$$\overline{T}_{2H}(p, z, s) = \frac{\cosh[z\psi(p, s)]}{\psi(p, s)\sinh[l\psi(p, s)]}\int_0^l F(p, \xi)\cosh[(l-\xi)\psi(p, s)]\,d\xi -$$

$$-\frac{1}{\psi(p, s)}\int_0^z F(p, \xi)\sinh[(z-\xi)\psi(p, s)]\,d\xi - \overline{A}_2(p, s)\frac{\cosh[(l-z)\psi(p, s)]}{\sinh[l\psi(p, s)]}, \quad 0 < z < l. \tag{21}$$

The formula of the reversal of the Hankel transform for expression (21) can be written in the form

$$\overline{T}_2(r, z, s) = \frac{2}{R^2}\overline{T}_{2H}(0, z, s) + \frac{2}{R^2}\sum_{n=1}^{\infty}\frac{J_0(p_n r)}{J_0^2(p_n R)}\overline{T}_{2H}(p_n, z, s), \quad 0 < r < R, \quad 0 < z < l, \tag{22}$$

in which $p_n R = \mu_n$ are the zeros of the Bessel function $J_1(\mu_n)$.

Expression (21) explicitly yields the following value needed to calculate the Hankel inverse transform (22):

$$\overline{T}_{2H}(0, z, s) = \frac{\cosh\left[z\sqrt{\frac{s}{a_2}}\right]}{\sqrt{\frac{s}{a_2}}\sinh\left[l\sqrt{\frac{s}{a_2}}\right]}\int_0^l F(0, \xi)\cosh\left[(l-\xi)\sqrt{\frac{s}{a_2}}\right]d\xi -$$

$$-\sqrt{\frac{a_2}{s}}\int_0^z F(0, \xi)\sinh\left[(z-\xi)\sqrt{\frac{s}{a_2}}\right]d\xi - \overline{A}_2(0, s)\frac{\cosh\left[(l-z)\sqrt{\frac{s}{a_2}}\right]}{\sinh\left[l\sqrt{\frac{s}{a_2}}\right]}, \quad 0 < z < l,$$

which makes it possible, by using notation (21), to write the solution in the region of Laplace transforms for the finite cylinder as

$$\overline{T}_2(r, z, s) = \frac{2\cosh\left[z\sqrt{\frac{s}{a_2}}\right]}{R^2\sqrt{\frac{s}{a_2}}\sinh\left[l\sqrt{\frac{s}{a_2}}\right]}\int_0^l F(0, \xi)\cosh\left[(l-\xi)\sqrt{\frac{s}{a_2}}\right]d\xi -$$

$$-\frac{2}{R^2}\sqrt{\frac{a_2}{s}}\int_0^z F(0, \xi)\sinh\left[(z-\xi)\sqrt{\frac{s}{a_2}}\right]d\xi - 2\overline{A}_2(0, s)\frac{\cosh\left[(l-z)\sqrt{\frac{s}{a_2}}\right]}{R^2\sinh\left[l\sqrt{\frac{s}{a_2}}\right]} +$$

$$+\frac{2}{R^2}\sum_{n=1}^{\infty}\frac{J_0(p_n r)}{J_0^2(p_n R)}\bar{T}_{2H}(p_n, z, s), \quad 0<r<R, \quad 0<z<l, \tag{23}$$

in which it is necessary to determine the unknown functions $\bar{A}_2(p_n, s)$ to satisfy the mixed boundary conditions (11)–(13).

Formulas (15) and (23) yield the corresponding expressions

$$\bar{T}_1(r, 0, s)=\frac{T_{10}}{s}+\int_0^{\infty}\bar{C}(\rho, s)\, J_0(\rho r)\,\rho d\rho, \quad r>0;$$

$$\frac{\partial \bar{T}_1(r, 0, s)}{\partial z}=-\int_0^{\infty}\bar{C}(\rho, s)\sqrt{\rho^2+\frac{s}{a_1}}\, J_0(\rho r)\,\rho d\rho, \quad r>0;$$

$$\bar{T}_2(r, 0, s)=\frac{2}{R^2\sqrt{\frac{s}{a_2}}\sinh\left[l\sqrt{\frac{s}{a_2}}\right]}\int_0^{l}F(0, \xi)\cosh\left[(l-\xi)\sqrt{\frac{s}{a_2}}\right]d\xi-\frac{2}{R^2}\bar{A}_2(0, s)\cot\left[l\sqrt{\frac{s}{a_2}}\right]-$$

$$-\frac{2}{R^2}\sum_{n=1}^{\infty}\frac{J_0(p_n r)}{J_0^2(p_n R)}\bar{A}_2(p_n, s)\cot[l\psi(p_n, s)]+$$

$$+\frac{2}{R^2}\sum_{n=1}^{\infty}\frac{J_0(p_n r)}{J_0^2(p_n R)\,\psi(p_n, s)\sinh[l\psi(p_n, s)]}\int_0^{l}F(p_n, \xi)\cosh[(l-\xi)\psi(p_n, s)]\,d\xi, \quad 0<r<R,$$

$$\frac{\partial \bar{T}_2(r, 0, s)}{\partial z}=\frac{2}{R^2}\sqrt{\frac{s}{a_2}}\bar{A}_2(0, s)\cot\left[l\sqrt{\frac{s}{a_2}}\right]+\frac{2}{R^2}\sum_{n=1}^{\infty}\frac{J_0(p_n r)}{J_0^2(p_n R)}\bar{A}_2(p_n, s)\,\psi(p_n, s), \quad 0<r<R,$$

making it possible to write down the system of equations resulting from the mixed boundary conditions (11)–(13):

$$\frac{T_{10}}{s}+\int_0^{\infty}\bar{C}(\rho, s)\, J_0(\rho r)\,\rho d\rho=\frac{2}{R^2\sqrt{\frac{s}{a_2}}\sinh\left[l\sqrt{\frac{s}{a_2}}\right]}\int_0^{l}F(0, \xi)\cosh\left[(l-\xi)\sqrt{\frac{s}{a_2}}\right]d\xi-$$

$$-\frac{2}{R^2}\bar{A}_2(0, s)\cot\left[l\sqrt{\frac{s}{a_2}}\right]-\frac{2}{R^2}\sum_{n=1}^{\infty}\frac{J_0(p_n r)}{J_0^2(p_n R)}\bar{A}_2(p_n, s)\cot[l\psi(p_n, s)]+$$

$$+\frac{2}{R^2}\sum_{n=1}^{\infty}\frac{J_0(p_n r)}{J_0^2(p_n R)\,\psi(p_n, s)\sinh[l\psi(p_n, s)]}\int_0^{l}F(p_n, \xi)\cosh[(l-\xi)\psi(p_n, s)]\,d\xi, \quad 0<r<R; \tag{24}$$

$$-K_\lambda \int_0^\infty \overline{C}(\rho, s)\sqrt{\rho^2+\frac{s}{a_1}}\, J_0(\rho r)\,\rho d\rho = \frac{2}{R^2}\sqrt{\frac{s}{a_2}}\,\overline{A}_2(0, s)\cot\left[l\sqrt{\frac{s}{a_2}}\right]+$$

$$+\frac{2}{R^2}\sum_{n=1}^{\infty}\frac{J_0(p_n r)}{J_0^2(p_n R)}\overline{A}_2(p_n, s)\,\psi(p_n, s)\,, \quad 0<r<R\,; \tag{25}$$

$$\int_0^\infty \overline{C}(\rho, s)\sqrt{\rho^2+\frac{s}{a_1}}\, J_0(\rho r)\,\rho d\rho = 0\,, \quad R<r<\infty\,, \tag{26}$$

and, moreover, according to [1], when $z = 0$, the following expression is valid:

$$\overline{A}_2(0, s) = \int_0^R f(r, 0)\, r dr\,, \tag{27}$$

in which $f(r, 0)$ is the value of the initial temperature of the contacting end-face surface of the finite cylinder.

Moreover, it is known (see, e.g., [5, p. 634]) that if a certain function $\overline{\Phi}(r, s)$ is representable, in the interval $0<r<R$, by the Fourier–Dini series

$$\overline{\Phi}(r, s) = \sum_{n=1}^{\infty} B_n J_0\left(\mu_n \frac{r}{R}\right), \tag{28}$$

in which μ_n are the positive zeros of the Bessel function of the first kind and first order, $J_1(\mu)$, then the coefficients of this series are calculated from the formula

$$B_n = \frac{2}{R^2 J_0^2(\mu_n)}\int_0^R \overline{\Phi}(r, s)\, J_0\left(\mu_n\frac{r}{R}\right) r dr\,. \tag{29}$$

With the use of formulas (27)–(29), Eq. (25) yields

$$K_\lambda^{-1}\,\psi\left(\frac{\mu_n}{R}, s\right)\overline{A}_2\left(\frac{\mu_n}{R}, s\right) =$$

$$= \int_0^R J_0\left(\frac{\mu_n}{R}r\right)\left[\frac{2}{R^2 K_\lambda}\sqrt{\frac{s}{a_2}}\int_0^R f(\rho, 0)\,\rho d\rho + \int_0^\infty \overline{C}(\rho, s)\sqrt{\rho^2+\frac{s}{a_1}}\, J_0(\rho r)\,\rho d\rho\right] r dr\,. \tag{30}$$

Since in our case

$$\int_0^R J_0\left(\frac{\mu_n}{R}r\right) r dr = \frac{R^2}{\mu_n} J_1(\mu_n) = 0\,, \quad \int_0^R J_0(pr)\, J_0\left(\frac{\mu_n}{R}r\right) r dr = \begin{cases} 0 \quad \text{for} \quad p \neq \dfrac{\mu_n}{R}\,, \\ \dfrac{R^2}{2} J_0^2(\mu_n) \quad \text{for} \quad p = \dfrac{\mu_n}{R} \end{cases}$$

by virtue of the orthogonality of the corresponding functions (see, e.g., [5]), from formula (30) the following expression can be written for the unknown function:

$$\overline{A}_2\left(\frac{\mu_n}{R}, s\right) = \frac{K_\lambda R^2}{2\psi\left(\frac{\mu_n}{R}, s\right)} J_0^2(\mu_n) \int_0^\infty \overline{C}(\rho, s)\sqrt{\rho^2 + \frac{s}{a_1}}\, \rho d\rho\,, \tag{31}$$

the substitution of which into Eq. (24) leads to the formula

$$\int_0^\infty \overline{C}(\rho, s)\, J_0(\rho r)\, \rho d\rho = -\frac{T_{10}}{s} - \frac{2}{R^2}\cot\left[l\sqrt{\frac{s}{a_2}}\right]\int_0^R f(\rho, 0)\, \rho d\rho +$$

$$+\frac{2}{R^2\sqrt{\frac{s}{a_2}}\sinh\left[l\sqrt{\frac{s}{a_2}}\right]}\int_0^l F(0, \xi)\cosh\left[(l-\xi)\sqrt{\frac{s}{a_2}}\right]d\xi +$$

$$+\frac{2}{R^2}\sum_{n=1}^\infty \frac{J_0\left(\frac{\mu_n}{R}r\right)}{J_0^2(\mu_n)\,\psi\left(\frac{\mu_n}{R}, s\right)\sinh\left[l\psi\left(\frac{\mu_n}{R}, s\right)\right]}\int_0^l F\left(\frac{\mu_n}{R}, \xi\right)\cosh\left[(l-\xi)\,\psi\left(\frac{\mu_n}{R}, s\right)\right]d\xi -$$

$$-\int_0^\infty \overline{C}(\rho, s)\sqrt{p^2 + \frac{s}{a_1}}\, pdp \sum_{n=1}^\infty \frac{K_\lambda \cot\left[l\psi\left(\frac{\mu_n}{R}, s\right)\right]}{\psi\left(\frac{\mu_n}{R}, s\right)} J_0\left(\frac{\mu_n}{R}r\right), \quad 0 < r < R\,. \tag{32}$$

Finally, using, on the right-hand side of Eq. (32), the equality proven in [3]

$$\sum_{n=1}^\infty \frac{K_\lambda \cot\left[l\psi\left(\frac{\mu_n}{R}, s\right)\right]}{\psi\left(\frac{\mu_n}{R}, s\right)} J_0\left(\frac{\mu_n}{R}r\right) = \frac{K_\lambda \cot\left[l\psi(p, s)\right]}{\psi(p,s)} J_0(pr)\,,$$

we arrive at the first integral equation with the L-parameter to find the unknown analytical function $\overline{C}(p, s)$ in the region $0 < r < R$, $z = 0$:

$$\int_0^\infty \overline{C}(p, s)\left(1 + K_\lambda \frac{\sqrt{p^2 + \frac{s}{a_1}}}{\sqrt{p^2 + \frac{s}{a_2}}}\cot\left[l\sqrt{p^2 + \frac{s}{a_2}}\right]\right) J_0(pr)\, pdp = -\frac{T_{10}}{s} + \overline{\Phi}(r, s)\,, \quad 0 < r < R\,, \tag{33}$$

$$\overline{\Phi}(r, s) = \frac{2}{R^2} \sum_{n=1}^{\infty} \frac{J_0\left(\frac{\mu_n}{R} r\right)}{J_0^2(\mu_n)\, \psi\left(\frac{\mu_n}{R}, s\right) \sinh\left[l\psi\left(\frac{\mu_n}{R}, s\right)\right]} \int_0^l F\left(\frac{\mu_n}{R}, \xi\right) \cosh\left[(l-\xi)\, \psi\left(\frac{\mu_n}{R}, s\right)\right] d\xi -$$

$$- \frac{2}{R^2} \cot\left[l\sqrt{\frac{s}{a_2}}\right] \int_0^R f(\rho, 0)\, \rho d\rho + \frac{2}{R^2 \sqrt{\frac{s}{a_2}} \sinh\left[l\sqrt{\frac{s}{a_2}}\right]} \int_0^l F(0, \xi) \cosh\left[(l-\xi)\sqrt{\frac{s}{a_2}}\right] d\xi\,.$$

In the region $R < r < \infty$, $z = 0$ the second integral equation with the L-parameter for finding the unknown analytical function $\overline{C}(p, s)$ has the form of Eq. (26):

$$\int_0^{\infty} \overline{C}(p, s) \sqrt{p^2 + \frac{s}{a_1}}\, J_0(pr)\, p dp = 0\,, \quad R < r < \infty\,. \tag{34}$$

To solve the paired integral equations (33), (34) the substitution

$$\overline{C}(p, s) \frac{1}{\sqrt{p^2 + \frac{s}{a_1}}} \int_0^R \overline{\varphi}(t, s) \cos\left[t\sqrt{p^2 + \frac{s}{a_1}}\right] dt\,, \tag{35}$$

is used, which ensures automatic fulfillment of the second paired equation (34) at any choice of the new unknown analytical function $\overline{\varphi}(t, s) = \overline{\varphi}(-t, s)$ owing to the value of the corresponding discontinuous integral (see [3, p. 489]) and from the first paired integral equation (33) provides the integral equation with the L-parameter

$$\int_0^R \overline{\varphi}(t, s) \int_0^{\infty} \frac{\cos\left[t\sqrt{p^2 + \frac{s}{a_1}}\right]}{\sqrt{p^2 + \frac{s}{a_1}}} J_0(pr)\, p dp\, dt +$$

$$+ K_\lambda \int_0^R \overline{\varphi}(t, s) \int_0^{\infty} \frac{\cot\left[l\sqrt{p^2 + \frac{s}{a_2}}\right]}{\sqrt{p^2 + \frac{s}{a_2}}} \cos\left[t\sqrt{p^2 + \frac{s}{a_1}}\right] J_0(pr)\, p dp\, dt = -\frac{T_{10}}{s} + \overline{\Phi}(r, s)\,, \quad 0 < r < R\,. \tag{36}$$

We note that the improper internal integrals on the left-hand side of Eq. (36) converge at any $0 < r < R$, $\mathrm{Re}\ s > 0$. In particular,

$$\int_0^\infty \frac{\cos\left[t\sqrt{p^2+\frac{s}{a_1}}\right]}{\sqrt{p^2+\frac{s}{a_1}}} J_0(pr)\,pdp = \begin{cases} \dfrac{\exp\left[-\sqrt{\frac{s}{a_1}(r^2-t^2)}\right]}{\sqrt{r^2-t^2}}, & 0<t<r, \\ \dfrac{\sin\left[-\sqrt{\frac{s}{a_1}(t^2-r^2)}\right]}{\sqrt{t^2-r^2}}, & 0<r<t, \end{cases}$$

$$\int_0^\infty \frac{\cot\left[l\sqrt{p^2+\frac{s}{a_2}}\right]}{\sqrt{p^2+\frac{s}{a_2}}} \cos\left[t\sqrt{p^2+\frac{s}{a_1}}\right] J_0(pr)\,pdp =$$

$$= \frac{1}{l}\int_{s^*}^\infty \cot[x]\cos\left[t\sqrt{\frac{x^2}{l^2}+s\left(\frac{a_2-a_1}{a_1a_2}\right)}\right] J_0\left(r\sqrt{\frac{x^2}{l^2}-\frac{s}{a_2}}\right)dx, \quad s^* = l\sqrt{\frac{s}{a_2}},$$

which allows one, just as in [3], to transform the integral equation (36) as

$$\int_0^r \frac{\bar\varphi(t,s)}{\sqrt{r^2-t^2}} \exp\left[-\sqrt{\frac{s}{a_1}(r^2-t^2)}\right]dt - \int_r^R \frac{\bar\varphi(t,s)}{\sqrt{t^2-r^2}} \sin\left[\sqrt{\frac{s}{a_1}(t^2-r^2)}\right]dt +$$

$$+\frac{K_\lambda}{l}\int_0^R \bar\varphi(t,s)\int_{s^*}^\infty \cot[x]\cos\left[t\sqrt{\frac{x^2}{l^2}+s\left(\frac{a_2-a_1}{a_1a_2}\right)}\right] J_0\left(r\sqrt{\frac{x^2}{l^2}-\frac{s}{a_2}}\right)dxdt = -\frac{T_{10}}{s}+\bar\Phi(r,s), \quad 0<r<R. \tag{37}$$

If the height of the cylinder $l \to \infty$, then Eq. (37) is the basis for determining the unknown auxiliary function in the case of the model of thermal contact of a semi-infinite cylinder and semispace.

If the radius of the cylinder $R \to \infty$ at a constant height of the cylinder l, i.e., a model of thermal contact of an infinitely long plate and semispace is considered, Eq. (37) does not appear, which is natural in the absence of mixed boundary conditions.

If simultaneously $R \to \infty$ and $l \to \infty$, we have a model of thermal contact of two semi-infinite bodies with different thermophysical properties and different initial temperatures (this case is considered in [1]).

Here, to illustrate the solution of the equations of the type (37) we will consider a simplified model of heat exchange at $f(r, z) = 0$, i.e., when $\Phi(r, s) = 0$ in Eqs. (33), (36), and (37).

To solve a corresponding equation of the form (37), we replace in it the variable r by μ, multiply both parts of the resulting equation by the integrating factor $2\mu\cos\left(\sqrt{(r^2-\mu^2)s/a_1}\right)/\sqrt{r^2-\mu^2}$ and integrate over μ from zero to r. Using the equality

$$\frac{d}{dr}\int_0^r \frac{2\cos\left[\sqrt{\frac{s}{a_1}}\sqrt{r^2-\mu^2}\right]}{\sqrt{r^2-\mu^2}} J_0\left(\mu\sqrt{\frac{x}{l^2}-\frac{s}{a_2}}\right)\mu d\mu = 2\cos\left[r\sqrt{\frac{x^2}{l^2}-\frac{(a_2-a_1)\,s}{a_1a_2}}\right],$$

we obtain an integral equation with the L-parameter written in a standard fashion [see [3]):

$$\bar{\varphi}(r, s) - \frac{1}{\pi}\int_0^R \bar{\varphi}(t, s)\, K(r, t, s)\, dt = -\frac{2T_{10}}{\pi s}\cos\left[r\sqrt{\frac{s}{a_1}}\right], \quad 0 < r < R,$$

in which

$$K(r, t, s) = \frac{\sin\left[(t-r)\sqrt{\frac{s}{a_1}}\right]}{t-r} + \frac{\sin\left[(t+r)\sqrt{\frac{s}{a_1}}\right]}{t+r} +$$

$$+\frac{K_\lambda}{l}\int_{s^*}^{\infty}\cot[x]\left(\cos\left[(t-r)\sqrt{\frac{x^2}{l^2}+s\left(\frac{a_2-a_1}{a_1a_2}\right)}\right] + \cos\left[(t+r)\sqrt{\frac{x^2}{l^2}+s\left(\frac{a_2-a_1}{a_1a_2}\right)}\right]\right)dx, \quad s^* = l\sqrt{\frac{s}{a_2}},$$

The methods of solution of this equation are considered in [3].

Thus, after determining the function $\bar{\varphi}(t, s)$, from formula (35) we find the unknown function $\bar{C}(p, s)$, the substitution of which into (15) makes it possible to determine the temperature field $\bar{T}_1(r, z, s)$ in the region of L-transforms for the semispace. Applying the inverse Laplace transform, we find the inverted transform $T_1(r, z, \tau)$. The temperature field $\bar{T}_2(r, z, s)$ in a finite cylinder is determined by formula (23) with the use of the value of $\bar{A}_2(\mu_n/R, s)$ from (31).

In conclusion, it should be noted that when $s \to 0$ ($\tau \to \infty$) the analytical results obtained describe stationary models of contact heat exchange between a finite cylinder and a semi-infinite body.

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